

# A SPECHT FILTRATION OF AN INDUCED SPECHT MODULE

ANDREW MATHAS

*To John Cannon and Derek Holt on the occasions of their significant birthdays,  
in recognition of their distinguished contributions to mathematics.*

**ABSTRACT.** Let  $\mathcal{H}_n$  be a (degenerate or non-degenerate) Hecke algebra of type  $G(\ell, 1, n)$ , defined over a commutative ring  $R$  with one, and let  $S(\mu)$  be a Specht module for  $\mathcal{H}_n$ . This paper shows that the induced Specht module  $S(\mu) \otimes_{\mathcal{H}_n} \mathcal{H}_{n+1}$  has an explicit Specht filtration.

## 1. INTRODUCTION

The Ariki-Koike algebras, and their rational degenerations, are interesting algebras which appear naturally in the representation theory of affine Hecke algebras, quantum groups, symmetric groups and general linear groups; see [14, 18] for details. They include as special cases the group algebras of the Coxeter groups of type  $A$  (the symmetric groups) and the Coxeter groups of type  $B$  (the hyperoctahedral groups).

Let  $\mathcal{H}_n$  be an Ariki-Koike algebra, or a degenerate cyclotomic Hecke algebra, of type  $G(\ell, 1, n)$ , for integers  $\ell, n \geq 1$ . For each multipartition  $\mu$  of  $n$  there is a **Specht module**  $S(\mu)$ , which is a right  $\mathcal{H}_n$ -module. (All of the undefined terms and notation, here and below, can be found in section 2.) When  $\mathcal{H}_n$  is semisimple the Specht modules give a complete set of pairwise non-isomorphic irreducible  $\mathcal{H}_n$ -modules as  $\mu$  runs through the multipartitions of  $n$ . In general, the Specht modules are not irreducible however every irreducible  $\mathcal{H}_n$ -module arises, in a unique way, as the simple head of some Specht module.

The Hecke algebra  $\mathcal{H}_n$  embeds into  $\mathcal{H}_{n+1}$  so there are natural induction and restriction functors,  $\text{Ind}$  and  $\text{Res}$ , between the categories of finite dimensional  $\mathcal{H}_n$ -modules and  $\mathcal{H}_{n+1}$ -modules. By [2, Proposition 1.9], in the Ariki-Koike case the restriction of the Specht module  $S(\mu)$  to  $\mathcal{H}_{n-1}$  has a Specht filtration of the form

$$0 = R_0 \subset R_1 \subset \cdots \subset R_r = \text{Res } S(\mu),$$

such that  $R_j/R_{j-1} \cong S(\mu - \rho_j)$ , where  $\rho_1 > \rho_2 > \cdots > \rho_r$  are the removable nodes of  $\mu$ . Consequently, if  $\mathcal{H}_{n+1}$  is semisimple then by Frobenius reciprocity

$$\text{Ind } S(\mu) \cong S(\mu \cup \alpha_1) \oplus \cdots \oplus S(\mu \cup \alpha_a),$$

where  $\alpha_1, \dots, \alpha_a$  are the addable nodes of  $\mu$ . This note generalizes this result to the case when  $\mathcal{H}_n$  is not necessarily semisimple. More precisely, we prove the following:

**Main Theorem.** *Suppose that  $\mathcal{H}_n$  is an Ariki-Koike algebra or a degenerate cyclotomic Hecke algebra of type  $G(\ell, 1, n)$  and let  $\mu$  be a multipartition of  $n$ . Then, as an  $\mathcal{H}_{n+1}$ -module, the induced module  $\text{Ind } S(\mu)$  has a filtration*

$$0 = I_0 \subset I_1 \subset \cdots \subset I_a = \text{Ind } S(\mu),$$

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such that  $I_j/I_{j-1} \cong S(\boldsymbol{\mu} \cup \alpha_j)$ , where  $\alpha_1 > \alpha_2 > \cdots > \alpha_a$  are the addable nodes of  $\boldsymbol{\mu}$ .

This result is part of the folklore for the representation theory of these algebras, however, we have been unable to find a proof of it in the literature when  $\ell > 1$ . If  $\ell = 1$  then our Main Theorem is an old result of James [12, §17] in the degenerate case (that is, for the symmetric group), and it can be deduced from [9, Theorem 7.4] in the non-degenerate case (the Hecke algebra of the symmetric group). We prove our Main Theorem by giving an explicit construction of  $\text{Ind } S(\boldsymbol{\lambda})$ ; see Corollary (3.7). Our argument is similar in spirit to that originally used by James [12] for the symmetric groups in that we identify the induced module as a quotient of the corresponding permutation module. Our approach, which uses cellular basis techniques, gives an explicit Specht filtration of the induced module; in contrast, James' approach is recursive.

Suppose now that  $\mathcal{H}_n$  is defined over a field of characteristic  $p \geq 0$ , or a suitable discrete valuation ring. Then by projecting onto the blocks of  $\mathcal{H}_n$  the induction functor  $\text{Ind}$  can be decomposed as a direct sum of subfunctors

$$\text{Ind} = \bigoplus_{i \in I} i\text{-Ind},$$

where  $I = \mathbb{Z}/p\mathbb{Z}$ , in the degenerate case, and  $I = \{q^a Q_s \mid a \in \mathbb{Z} \text{ and } 1 \leq s \leq r\}$  in the non-degenerate case. (If the parameters  $Q_1, \dots, Q_r$  are all non-zero then, up to Morita equivalence, it is enough to consider the cases where  $Q_1, \dots, Q_r$  are all powers of  $q$  by the main result of [11]. In this case we can take  $I = \mathbb{Z}/e\mathbb{Z}$  where  $e$  is the smallest positive integer such that  $1 + q + \cdots + q^{e-1} = 0$ .) The functor  $i\text{-Ind}$  is a natural generalization of Robinson's  $i$ -induction functor; see [2, 1.11] and [14, §8] for the precise definitions.

**(1.1). Corollary.** *Suppose that  $\boldsymbol{\mu}$  is a multipartition of  $n$  and  $i \in I$ . Then  $i\text{-Ind } S(\boldsymbol{\mu})$  has a filtration*

$$0 = I_0 \subset I_1 \subset \cdots \subset I_b = i\text{-Ind } S(\boldsymbol{\mu}),$$

such that  $I_j/I_{j-1} \cong S(\boldsymbol{\mu} \cup \alpha_j)$ , where  $\alpha_1 > \alpha_2 > \cdots > \alpha_b$  are the addable  $i$ -nodes of  $\boldsymbol{\mu}$ .

*Proof.* By [15] and [4], the Specht modules  $S(\boldsymbol{\mu} \cup \alpha)$  and  $S(\boldsymbol{\mu} \cup \beta)$  are in the same block if and only if  $\alpha$  and  $\beta$  have the same residue. By the Main Theorem and the definition of the functor  $i\text{-Ind}$ , the Specht module  $S(\boldsymbol{\mu} \cup \alpha)$  is a subquotient of  $i\text{-Ind } S(\boldsymbol{\mu})$  if and only if  $\alpha$  is an  $i$ -node (cf. [2, Cor. 1.12]). This implies the result.  $\square$

Recently Brundan and Kleshchev [5] have shown that  $\mathcal{H}_n$  is naturally  $\mathbb{Z}$ -graded and Brundan, Kleshchev and Wang [8] have shown that  $S(\boldsymbol{\mu})$  admits a natural grading. There should be a graded analogue of our induction theorem; see [8, Remark 4.12] for a precise conjecture. Unfortunately, the arguments of this paper do not automatically lift to the graded setting because it is not clear how to use our results to find a homogeneous basis of the induced module.

## 2. ARIKI-KOIKE ALGEBRAS

In order to make this note self-contained, this section quickly recalls the definitions and results that we need from the literature and, at the same time, sets our notation. We concentrate on the non-degenerate case as the degenerate case follows in exactly the same way, with only minor changes of notation, using the results of [3, §6]. See the remarks at the end of this section for more details.

Throughout this note we fix positive integers  $\ell$  and  $n$  and let  $\mathfrak{S}_n$  be the symmetric group of degree  $n$ . For  $1 \leq i < n$  let  $s_i = (i, i+1) \in \mathfrak{S}_n$ . Then  $s_1, \dots, s_{n-1}$  are the standard Coxeter generators of  $\mathfrak{S}_n$ .

Let  $R$  be a commutative ring with 1 and let  $q, Q_1, \dots, Q_\ell$  be elements of  $R$  with  $q$  invertible. The Ariki–Koike algebra  $\mathcal{H}_n = \mathcal{H}_{R, \ell, n}(q, Q_1, \dots, Q_\ell)$  is the associative unital  $R$ -algebra with generators  $T_0, T_1, \dots, T_{n-1}$  and relations

$$\begin{aligned} (T_0 - Q_1) \dots (T_0 - Q_\ell) &= 0, \\ (T_i - q)(T_i + 1) &= 0, & \text{for } 1 \leq i \leq n-1, \\ T_0 T_1 T_0 T_1 &= T_1 T_0 T_1 T_0, \\ T_{i+1} T_i T_{i+1} &= T_i T_{i+1} T_i, & \text{for } 1 \leq i \leq n-2, \\ T_i T_j &= T_j T_i, & \text{for } 0 \leq i < j-1 \leq n-2. \end{aligned}$$

Using the relations it follows that there is a unique anti-isomorphism  $*$ :  $\mathcal{H}_n \longrightarrow \mathcal{H}_n$  such that  $T_i^* = T_i$ , for  $0 \leq i < n$ .

Ariki and Koike [1, Theorem 3.10] showed that  $\mathcal{H}_n$  is free as an  $R$ -module with basis  $\{L_1^{a_1} \dots L_n^{a_n} T_w \mid 0 \leq a_1, \dots, a_n < \ell \text{ and } w \in \mathfrak{S}_n\}$  where  $L_1 = T_0$  and  $L_{i+1} = q^{-1} T_i L_i T_i$  for  $i = 1, \dots, n-1$ , and  $T_w = T_{i_1} \dots T_{i_k}$  if  $w = s_{i_1} \dots s_{i_k} \in \mathfrak{S}_n$  is a reduced expression (that is,  $k$  is minimal).

The Ariki–Koike basis theorem implies that there is a natural embedding of  $\mathcal{H}_n$  in  $\mathcal{H}_{n+1}$  and that  $\mathcal{H}_{n+1}$  is free as an  $\mathcal{H}_n$ -module of rank  $\ell(n+1)$ . If  $M$  is an  $\mathcal{H}_n$ -module let

$$\text{Ind } M = M \otimes_{\mathcal{H}_n} \mathcal{H}_{n+1}$$

be the corresponding induced  $\mathcal{H}_{n+1}$ -module. Note that induction is an exact functor since  $\mathcal{H}_{n+1}$  is free as an  $\mathcal{H}_n$ -module.

We will need to the following easily proved property of the basis elements [10, 2.1].

**(2.1).** *Suppose that  $1 \leq k \leq n$ ,  $a \in R$  and  $w \in \mathfrak{S}_k \times \mathfrak{S}_{n-k}$ . Then*

$$(L_1 - a) \dots (L_k - a) T_w = T_w (L_1 - a) \dots (L_k - a).$$

The algebra  $\mathcal{H}_n$  has another basis which is crucial to this note. In order to describe it recall that a partition of  $n$  is a weakly decreasing sequence  $\lambda = (\lambda_1 \geq \lambda_2 \geq \dots)$  of non-negative integers such that  $|\lambda| = \sum_i \lambda_i = n$ . A **multipartition**, or  $\ell$ -partition, of  $n$  is an ordered  $\ell$ -tuple  $\lambda = (\lambda^{(1)}, \dots, \lambda^{(\ell)})$  of partitions such that  $|\lambda| = |\lambda^{(1)}| + \dots + |\lambda^{(\ell)}| = n$ . Let  $\Lambda_{\ell, n}^+$  be the set of multipartitions of  $n$ . If  $\lambda, \mu \in \Lambda_{\ell, n}^+$  then  $\lambda$  **dominates**  $\mu$ , and we write  $\lambda \succeq \mu$ , if

$$\sum_{t=1}^{s-1} |\lambda^{(t)}| + \sum_{i=1}^k \lambda_i^{(s)} \geq \sum_{t=1}^{s-1} |\mu^{(t)}| + \sum_{i=1}^k \mu_i^{(s)},$$

for  $1 \leq s \leq \ell$  and for all  $k \geq 1$ . Dominance is a partial order on  $\Lambda_{\ell, n}^+$ .

If  $\lambda$  is a multipartition let  $\mathfrak{S}_\lambda = \mathfrak{S}_{\lambda^{(1)}} \times \dots \times \mathfrak{S}_{\lambda^{(\ell)}}$  be the corresponding parabolic subgroup of  $\mathfrak{S}_n$  and set  $a_s^\lambda = \sum_{t=1}^{s-1} |\lambda^{(t)}|$ , for  $1 \leq s \leq \ell$ , and put  $a_{\ell+1}^\lambda = n - 1$ . Define  $m_\lambda = u_\lambda^+ x_\lambda$  where

$$u_\lambda^+ = \prod_{s=2}^{\ell} \prod_{k=1}^{a_s^\lambda} (L_k - Q_s) \quad \text{and} \quad x_\lambda = \sum_{w \in \mathfrak{S}_\lambda} T_w.$$

Then  $u_\lambda^+ x_\lambda = m_\lambda = x_\lambda u_\lambda^+$  by (2.1).

Let  $\lambda$  be a multipartition (of  $n$ ). The **diagram** of  $\lambda$  is the set of nodes

$$[\lambda] = \{ (r, c, s) \mid 1 \leq \lambda_r^{(s)} \leq c \text{ and } 1 \leq s \leq \ell \}.$$

More generally a **node** is any element of  $\mathbb{N} \times \mathbb{N} \times \{1, \dots, \ell\}$ , which we consider as a partially ordered set where  $(r, c, s) \geq (r', c', s')$  if either  $s > s'$ , or  $s = s'$  and  $r < r'$ . For the sake of Corollary (1.1) only, define the **residue** of the node  $(r, c, s)$  to be  $q^{c-r} Q_s$ .

An **addable** node of  $\lambda$  is any node  $\alpha \notin [\lambda]$  such that  $[\lambda] \cup \{\alpha\}$  is the diagram of some multipartition. Let  $\lambda \cup \alpha$  be the multipartition such that  $[\lambda \cup \alpha] = [\lambda] \cup \{\alpha\}$ . Similarly, a **removable** node of  $\lambda$  is a node  $\rho \in [\lambda]$  such that  $[\lambda] - \{\rho\}$  is the diagram of a multipartition; let  $\lambda - \rho$  be this multipartition. Note that the set of addable and removable nodes for  $\lambda$  are both totally ordered by  $>$ .

If  $X$  is a set then an  $X$ -valued  $\lambda$ -tableau is a function  $T : [\lambda] \rightarrow X$ . If  $T$  is a  $\lambda$ -tableau then we write  $\text{Shape}(T) = \lambda$ . For convenience we identify  $T = (T^{(1)}, \dots, T^{(\ell)})$  with a labeling of the diagram  $[\lambda]$  by elements of  $X$  in the obvious way. Thus, we can talk of the rows, columns and components of  $T$ .

A **standard  $\lambda$ -tableau** is a map  $t : [\lambda] \rightarrow \{1, 2, \dots, n\}$  such that for  $s = 1, \dots, \ell$  the entries in each row of  $t^{(s)}$  increase from left to right and the entries in each column of  $t^{(s)}$  increase from top to bottom. Let  $\mathcal{T}^{\text{Std}}(\lambda)$  be the set of standard  $\lambda$ -tableaux.

Let  $t^\lambda$  be the standard  $\lambda$ -tableau such that the entries in  $t^\lambda$  increase from left to right along the rows of  $t^{\lambda^{(1)}}, \dots, t^{\lambda^{(\ell)}}$  in order. If  $t$  is a standard  $\lambda$ -tableau let  $d(t) \in \mathfrak{S}_n$  be the unique permutation such that  $t = t^\lambda d(t)$ . Define  $m_{st} = T_{d(s)}^* m_\lambda T_{d(t)}$ , for  $s, t \in \mathcal{T}^{\text{Std}}(\lambda)$ . By [10, Theorem 3.26], the set

$$\{ m_{st} \mid s, t \in \mathcal{T}^{\text{Std}}(\lambda) \text{ and } \lambda \in \Lambda_{\ell, n}^+ \}$$

is a cellular basis of  $\mathcal{H}_n$ . Consequently, if  $\mathcal{H}_n(\lambda)$  is the  $R$ -module spanned by

$$\{ m_{st} \mid s, t \in \mathcal{T}^{\text{Std}}(\mu) \text{ for some } \mu \in \Lambda_{\ell, n}^+ \text{ with } \mu \triangleright \lambda \},$$

then  $\mathcal{H}_n(\lambda)$  is a two-sided ideal of  $\mathcal{H}_n$ .

The **Specht module**  $S(\lambda)$  is the submodule of  $\mathcal{H}_n / \mathcal{H}_n(\lambda)$  generated by  $m_\lambda + \mathcal{H}_n(\lambda)$ . It follows from the general theory of cellular algebras that  $S(\lambda)$  is free as an  $R$ -module with basis  $\{ m_t \mid t \in \mathcal{T}^{\text{Std}}(\lambda) \}$ , where  $m_t = m_{\mu_t} + \mathcal{H}_n(\lambda)$  for  $t \in \mathcal{T}^{\text{Std}}(\lambda)$ .

Let  $M$  be an  $\mathcal{H}_n$ -module. Then  $M$  has a **Specht filtration** if there exists a filtration

$$0 = M_0 \subset M_1 \subset \dots \subset M_k = M$$

and multipartitions  $\lambda_1, \dots, \lambda_k$  such that  $M_i / M_{i-1} \cong S(\lambda_i)$ , for  $i = 1, \dots, k$ .

For each multipartition  $\mu \in \Lambda_{\ell, n}^+$  let  $M(\mu) = m_\mu \mathcal{H}_n$ . The final result that we will need gives an explicit Specht filtration of  $M(\mu)$ . The proof of our Main Theorem is inspired by this filtration.

Given two tuples  $(i, s)$  and  $(j, t)$  write  $(i, s) \preceq (j, t)$  if either  $s < t$ , or  $s = t$  and  $i \leq j$ .

**(2.2). Definition ([10, Definition 4.4]).** Suppose that  $\lambda, \mu \in \Lambda_{\ell, n}^+$  and let  $T : [\lambda] \rightarrow \mathbb{N} \times \{1, 2, \dots, \ell\}$  be a  $\lambda$ -tableau. Then:

- a)  $T$  is a tableau of **type  $\mu$**  if  $\mu_i^{(s)} = \# \{ x \in [\lambda] \mid T(x) = (i, s) \}$ , for all  $i \geq 1$  and  $1 \leq s \leq \ell$ .
- b)  $T$  is **semistandard** if the entries in each component  $T^{(s)}$ , for  $1 \leq s \leq \ell$ , of  $T$  are:
  - i) weakly increasing from left to right along each row (with respect to  $\preceq$ );
  - ii) strictly increasing from top to bottom down columns; and,
  - iii)  $(j, t)$  appears in  $T^{(s)}$  only if  $t \geq s$ .

Let  $\mathcal{T}_\mu^{\text{SSStd}}(\lambda)$  be the set of semistandard  $\lambda$ -tableaux of type  $\mu$  and let  $\mathcal{T}_\mu^{\text{SSStd}}(\Lambda_{\ell, n}^+) = \bigcup_{\lambda \in \Lambda_{\ell, n}^+} \mathcal{T}_\mu^{\text{SSStd}}(\lambda)$  be the set of all semistandard tableaux of type  $\mu$ .

Let  $t$  be a standard  $\lambda$ -tableau. Define  $\mu(t)$  to be the tableau obtained from  $t$  by replacing each entry  $j$  in  $t$  with  $(i, s)$  if  $j$  appears in row  $i$  of  $t^{\mu^{(s)}}$ . The tableau  $\mu(t)$  is a  $\lambda$ -tableau

of type  $\mu$ ; it is not necessarily semistandard. Finally, if  $S \in \mathcal{T}_\mu^{\text{Std}}(\lambda)$  and  $t \in \mathcal{T}^{\text{Std}}(\lambda)$  set

$$m_{St} = \sum_{\substack{s \in \mathcal{T}^{\text{Std}}(\lambda) \\ \mu(s)=S}} m_{st}.$$

**(2.3)** ([10, Theorem 4.14 and Corollary 4.15]). *Suppose that  $\lambda, \mu \in \Lambda_{\ell,n}^+$ . Then:*

a)  $M(\mu)$  is free as an  $R$ -module with basis

$$\{ m_{St} \mid S \in \mathcal{T}_\mu^{\text{Std}}(\lambda), t \in \mathcal{T}^{\text{Std}}(\lambda) \text{ for some } \lambda \in \Lambda_{\ell,n}^+ \}.$$

b) *Suppose that  $\mathcal{T}_\mu^{\text{Std}}(\Lambda_{\ell,n}^+) = \{S_1, \dots, S_m\}$  ordered so that  $i \leq j$  whenever  $\lambda_i \geq \lambda_j$ , where  $\lambda_i = \text{Shape}(S_i)$ . Let  $M_i$  be the  $R$ -submodule of  $M(\mu)$  spanned by the elements  $\{ m_{S_j t} \mid j \leq i \text{ and } t \in \mathcal{T}^{\text{Std}}(\lambda_j) \}$ , Then*

$$0 = M_0 \subset M_1 \subset \dots \subset M_m = M(\mu)$$

*is an  $\mathcal{H}_n$ -module filtration of  $M(\mu)$  and  $M_i/M_{i-1} \cong S(\lambda_i)$ , for  $1 \leq i \leq m$ .*

**(2.4). Remark.** Very few changes need to be made to the results above in the degenerate case. The analogue of the cellular basis  $\{m_{st}\}$  in the degenerate case is constructed in [3, §6]. Using this basis of the degenerate Hecke algebra, the construction of the Specht filtration of the ideals  $M(\mu)$  follows easily using the arguments of [10, §4]; cf. [7, Cor. 6.13]. The arguments in the next section, modulo minor differences in the meaning of the symbols, applies to both the degenerate and non-degenerate cases.

### 3. INDUCING SPECHT MODULES

We are now ready to start proving the Main Theorem. Fix a multipartition  $\mu \in \Lambda_{\ell,n}^+$ . As in (2.3) we let  $\mathcal{T}_\mu^{\text{Std}}(\Lambda_{\ell,n}^+) = \{S_1, \dots, S_m\}$  be the set of semistandard tableau of type  $\mu$  ordered so that  $i \leq j$  whenever  $\lambda_i \geq \lambda_j$ , where  $\lambda_i = \text{Shape}(S_i)$  for  $1 \leq i \leq m$ . So, in particular,  $S_m = T^\mu = \mu(t^\mu)$  is the unique semistandard  $\mu$ -tableaux of type  $\mu$ .

Throughout this section we freely identify  $\mathcal{H}_n$  with its image under the natural embedding  $\mathcal{H}_n \hookrightarrow \mathcal{H}_{n+1}$ . In particular, we will think of the basis element  $m_{st}$  as an element of  $\mathcal{H}_{n+1}$ , for standard  $\lambda$ -tableaux  $s, t \in \mathcal{T}^{\text{Std}}(\lambda)$  with  $\lambda \in \Lambda_{\ell,n}^+$ . This embedding also identifies  $\text{Ind } M(\mu)$  with a submodule of  $\mathcal{H}_{n+1}$ .

The following simple Lemma contains the idea which drives our proof.

**(3.1). Lemma.** *Suppose that  $\mu$  is a multipartition of  $n$  and let  $\omega$  be the lowest addable node of  $\mu$  (that is,  $\alpha \geq \omega$  whenever  $\alpha$  is an addable node of  $\mu$ ). Then :*

- a)  $\text{Ind } M(\mu) = M(\mu \cup \omega)$ .
- b) *The induced module  $\text{Ind } M(\mu)$  has a filtration*

$$0 = N_0 \subset N_1 \subset \dots \subset N_m = \text{Ind } M(\mu)$$

*such that  $N_i/N_{i-1} \cong \text{Ind } S(\lambda_i)$ , where  $\lambda_i = \text{Shape}(S_i)$  for  $1 \leq i \leq m$ .*

*Proof.* By definition,  $m_\mu = m_{\mu \cup \omega}$  using the embedding  $\mathcal{H}_n \hookrightarrow \mathcal{H}_{n+1}$ . Therefore,

$$\text{Ind } M(\mu) = m_\mu \mathcal{H}_n \otimes_{\mathcal{H}_n} \mathcal{H}_{n+1} = m_\mu \mathcal{H}_{n+1} = m_{\mu \cup \omega} \mathcal{H}_{n+1} = M(\mu \cup \omega),$$

proving (a). As induction is exact, part (b) follows from part (a) and (2.3)(b).  $\square$

If  $\mu = ((n), (0), \dots, (0))$  then  $S(\mu) = M(\mu)$ . The Main Theorem in this special case is just part (b) of the Lemma. To prove the theorem when  $\mu \neq ((n), (0), \dots, (0))$  we explicitly describe the filtration of  $\text{Ind } M(\mu)$  given by the Lemma in terms of the basis of  $M(\mu \cup \omega)$  from (2.3).

Let  $\omega$  be the lowest addable node of  $\mu$ . Then  $\omega = (z, 1, \ell)$ , where  $z \geq 1$  is minimal such that  $(z, 1, \ell) \notin [\mu]$ . Suppose that  $S \in \mathcal{T}_{\mu}^{\text{Std}}(\lambda)$ , for some  $\lambda \in \Lambda_{\ell, n}^+$ , and that  $\beta$  is an addable node of  $\lambda$ . Let  $S \cup \beta$  be the semistandard  $(\lambda \cup \beta)$ -tableau given by

$$(S \cup \beta)(\eta) = \begin{cases} S(\eta), & \text{if } \eta \in [\lambda], \\ (z, \ell), & \text{if } \eta = \beta. \end{cases}$$

Thus  $S \cup \beta$  is the semistandard  $(\lambda \cup \beta)$ -tableau of type  $\mu \cup \omega$  obtained by adding the node  $\beta$  to  $S$  with label  $(z, \ell)$ . Let  $\mathcal{T}_{\mu \cup \omega}^{\text{Std}}(S)$  be the set of semistandard tableau of type  $\mu \cup \omega$  obtained in this way from  $S$  as  $\beta$  runs over the addable nodes of  $\lambda$ . It is easy to see that every semistandard tableau of type  $\mu \cup \omega$  arises uniquely in this way, so

$$(3.2) \quad \mathcal{T}_{\mu \cup \omega}^{\text{Std}}(\Lambda_{\ell, n+1}^+) = \coprod_{S \in \mathcal{T}_{\mu}^{\text{Std}}(\Lambda_{\ell, n}^+)} \mathcal{T}_{\mu \cup \omega}^{\text{Std}}(S).$$

Armed with this notation, observe that if  $S \in \mathcal{T}_{\mu}^{\text{Std}}(\lambda)$  then  $m_{S \cup \beta} = m_{(S \cup \beta) \cup \beta}$ , as an element of  $\mathcal{H}_{n+1}$ , where  $\beta$  is the lowest addable node of  $\lambda$ .

Suppose that  $1 \leq a \leq b < n$ . Let  $\mathfrak{S}_{a,b}$  be the symmetric group on  $\{a, a+1, \dots, b\}$  and set  $s_{b,a} = (b, b+1) \dots (a, a+1) \in \mathfrak{S}_n$  and  $T_{b,a} = T_{s_{b,a}} = T_b \dots T_a$ . For convenience, we set  $T_{b,a} = 1$  if  $b < a$ . The following useful identity is surely known.

**(3.3). Lemma.** *Suppose that  $1 \leq a < b \leq n$ . Then*

$$\left( \sum_{w \in \mathfrak{S}_{a,b}} T_w \right) T_{b,a} = T_{b,a} \left( \sum_{v \in \mathfrak{S}_{a+1,b+1}} T_v \right).$$

*Proof.* It is easy to check that  $\mathfrak{S}_{a,b} s_{b,a} = s_{b,a} \mathfrak{S}_{a+1,b+1}$  and that  $s_{b,a}$  is a distinguished  $(\mathfrak{S}_{a,b}, \mathfrak{S}_{a+1,b+1})$ -double coset representative (in the sense of [16, Prop. 4.4], for example). Therefore, if  $w \in \mathfrak{S}_{a,b}$  and  $v = s_{b,a} w s_{b,a} \in \mathfrak{S}_{a+1,b+1}$  then  $T_w T_{b,a} = T_{w s_{b,a}} = T_{s_{b,a} v} = T_{b,a} T_v$  by [16, Prop. 3.3]. This implies the Lemma.  $\square$

**(3.4). Lemma.** *Suppose that  $\lambda \in \Lambda_{\ell, n}^+$  and  $\nu = \lambda \cup \beta$ , where  $\beta = (r, c, e)$  is an addable node of  $\lambda$ . Then  $T_{n-1, a+1} m_{\nu} \in m_{\lambda} \mathcal{H}_{n+1}$ , where  $a = a_1^{\lambda} + \dots + a_e^{\lambda} + \lambda_1^{(e)} + \dots + \lambda_r^{(e)}$ .*

*Proof.* Let  $D_{d,a} = 1 + T_a + T_{a,a-1} + \dots + T_{a,d}$ , where  $d = a - \lambda_r^{(e)} + 1$ . Then  $D_{d,a}$  is the sum of distinguished right coset representatives for  $\mathfrak{S}_{d,a}$  in  $\mathfrak{S}_{d,a+1}$ . Therefore,  $x_{\lambda} T_{n-1, a+1} D_{d,a} = T_{n-1, a+1} x_{\nu}$  by Lemma (3.3). On the other hand, it follows directly from the definitions that  $u_{\nu}^+ = u_{\lambda}^+ (L_{a_{\ell}^{\lambda}+1} - Q_{\ell}) \dots (L_{a_{e+1}^{\lambda}+1} - Q_{e+1})$ . Therefore, writing  $m_{\lambda} = x_{\lambda} u_{\lambda}^+$  and using (2.1) we see that

$$m_{\lambda} \left( \prod_{s=\ell, \dots, e+1} T_{a_{s+1}^{\lambda}, a_s^{\lambda}+1} (L_{a_s^{\lambda}+1} - Q_s) \right) T_{a_{e+1}^{\lambda}, a+1} D_{d,a} = T_{n-1, a+1} m_{\nu},$$

where the product on the left-hand side is read in order, from left to right, with decreasing values of  $s$ . (Recall that, for convenience,  $a_{\ell+1}^{\lambda} = n-1$  and  $T_{n-1, n} = 1$ .)  $\square$

Let  $\leq$  be the Bruhat order on  $\mathfrak{S}_n$ ; see, for example, [16, p.30]. If  $S$  is a semistandard  $\lambda$ -tableau of type  $\mu$  let  $\dot{S}$  be the unique standard  $\lambda$ -tableau such that  $\mu(\dot{S}) = S$  and  $d(\dot{S}) \leq d(\mathfrak{s})$  whenever  $\mathfrak{s} \in \mathcal{T}^{\text{Std}}(\lambda)$  and  $\mu(\mathfrak{s}) = S$ . Such a tableau  $\dot{S}$  exists by [13, Lemma 3.9].

**(3.5). Lemma.** *Suppose that  $S \in \mathcal{T}_{\mu}^{\text{Std}}(\lambda)$  and that  $U \in \mathcal{T}_{\mu \cup \omega}^{\text{Std}}(S)$ . Let  $\nu = \text{Shape}(U)$ . Then  $m_{U \cup \nu} \in m_{S \cup \lambda} \mathcal{H}_{n+1}$ .*

*Proof.* Definition,  $m_{S_t\lambda} = \sum_s m_{s_t\lambda}$  where  $d(s)$  runs over a set of right  $\mathfrak{S}_\mu$ -coset representatives in the double coset  $\mathfrak{S}_\lambda d(\dot{S})\mathfrak{S}_\mu$ . Therefore,  $m_{S_t\lambda} = h_S T_{d(\dot{S})}^* m_\lambda$  for some  $h_S \in \mathcal{H}_q(\mathfrak{S}_\mu)$ . (Explicitly,  $h_S = \sum_d T_d$  where  $d$  runs over the set of distinguished left coset representatives of  $\mathfrak{S}_\mu \cap d(\dot{S})\mathfrak{S}_\lambda d(\dot{S})^{-1}$  in  $\mathfrak{S}_\mu$ .)

As in Lemma (3.4), write  $\nu = \lambda \cup \beta$ , where  $\beta = (r, c, e)$  and set  $a = a_1^\lambda + \dots + a_e^\lambda + \lambda_1^{(e)} + \dots + \lambda_r^{(e)}$ . Then  $U = S \cup \beta$ . Therefore,  $d(\dot{U}) = s_{n-1, a+1}^{-1} d(\dot{S})$ , so that  $m_{U\nu} = h T_{d(\dot{S})}^* T_{n-1, a+1} m_\nu$ .

Finally,  $T_{n-1, a+1} m_\nu = m_\lambda h_{\nu, a}$ , for some  $h_{\nu, a} \in \mathcal{H}_{n+1}$ , by Lemma (3.4). Therefore,

$$m_{U\nu} = h_S T_{d(\dot{S})}^* T_{n-1, a+1} m_\nu = h_S T_{d(\dot{S})}^* m_\lambda h_{\nu, a} = m_{S_t\lambda} h_{\nu, a} \in m_{S_t\lambda} \mathcal{H}_{n+1},$$

as required.  $\square$

We can now make the filtration of Lemma (3.1)(b) explicit. As a result we will show that we can obtain a basis for the induced module by adding a node labeled  $(z, \ell)$  to the basis elements of  $M(\mu)$  in all possible ways.

**(3.6). Theorem.** Suppose that  $\mu \in \Lambda_{\ell, n}^+$  and order  $\mathcal{T}_\mu^{SStd}(\Lambda_{\ell, n}^+) = \{S_1, \dots, S_m\}$  as above, with  $\lambda_i = \text{Shape}(S_i)$ . Let  $N_i$  be the  $R$ -submodule of  $M(\mu \cup \omega)$  spanned by the elements

$$\{m_{U\nu} \mid U \in \mathcal{T}_{\mu \cup \omega}^{SStd}(S_j), \nu \in \mathcal{T}^{Std}(\text{Shape}(U)) \text{ for } 1 \leq j \leq i\},$$

for  $i = 0, 1, \dots, m$ . Then  $N_i$  is an  $\mathcal{H}_{n+1}$ -submodule of  $\text{Ind } M(\lambda)$  and

$$\text{Ind } S(\lambda_i) \cong N_i / N_{i-1},$$

for  $1 \leq i \leq m$ .

*Proof.* By Lemma (3.1)(a),  $\text{Ind } M(\mu) = M(\mu \cup \omega)$  and by (3.2) the set of elements

$$\{m_{U\nu} \mid U \in \mathcal{T}_{\mu \cup \omega}^{SStd}(S_j), \nu \in \mathcal{T}^{Std}(\text{Shape}(U)) \text{ for } 1 \leq j \leq m\}$$

is precisely the basis of  $M(\mu \cup \omega)$  given by (2.3), so  $M(\mu \cup \omega) = N_m$ . Moreover, since  $\mathcal{H}_{n+1}(\nu)$  is a two-sided ideal of  $\mathcal{H}_{n+1}$  for all  $\nu \in \Lambda_{\ell, n+1}^+$ , the action of  $\mathcal{H}_{n+1}$  on the basis  $\{m_{U\nu}\}$  respects dominance, so  $N_i$  is a submodule of  $M(\mu \cup \omega)$ , for  $0 \leq i \leq m$ .

Recall the filtration  $0 = M_0 \subset M_1 \subset \dots \subset M_m = M(\lambda)$  of  $M(\lambda)$  given in (2.3). By Lemma (3.1)(b), to prove the Theorem it is enough to show by induction on  $i$  that  $\text{Ind } M_i = N_i$ , for  $0 \leq i < m$ . This is trivially true when  $i = 0$  so we may assume that  $i > 0$ .

To show that  $\text{Ind } M_i \subseteq N_i$  note that  $m_{S_i t \lambda_i} \in N_i$  and  $m_{S_i t \lambda_i} + M_{i-1}$  generates  $M_i / M_{i-1}$  as an  $\mathcal{H}_n$ -module. Therefore,  $\text{Ind } M_i \subseteq N_i$  by induction on  $i$ .

To prove the reverse inclusion, suppose that  $U \in \mathcal{T}_{\mu \cup \omega}^{SStd}(S_i)$  and let  $\nu = \text{Shape}(U)$ . Then  $m_{U\nu} \in m_{S_i t \lambda_i} \mathcal{H}_{n+1} \subseteq \text{Ind } M_i$  by Lemma (3.5). Therefore,  $m_{U\nu} \in \text{Ind } M_i$ , for any  $\nu \in \mathcal{T}^{Std}(\nu)$ . It follows by induction that  $N_i \subseteq \text{Ind } M_i$  as required.  $\square$

For each addable node  $\beta$  of  $\mu$  let  $N^\beta$  be the submodule of  $M(\mu \cup \omega)$  spanned by

$$\{m_{U\nu} \mid U \in \mathcal{T}_{\mu \cup \omega}^{SStd}(\lambda), \nu \in \mathcal{T}^{Std}(\lambda) \text{ where } \lambda \in \Lambda_{\ell, n+1}^+ \text{ and } \lambda \triangleright \mu \cup \beta\} + N_{m-1},$$

where  $N_{m-1}$  is the submodule of  $M(\mu \cup \omega)$  defined in Theorem (3.6). Note, in particular, that  $N^\alpha = N_{m-1}$ .

We can now prove a more explicit version of the Main Theorem of this paper.

**(3.7). Corollary.** *Suppose that  $\mu$  is a multipartition of  $n$  and let  $\alpha_1 = \alpha > \cdots > \alpha_a = \omega$  be the addable nodes of  $\mu$ . Then  $\text{Ind } S(\mu) \cong M(\mu \cup \omega)/N^\alpha$  is a free  $R$ -module with basis*

$$\{ m_{\mathbf{U}\mathbf{v}} + N^\alpha \mid \mathbf{U} \in \mathcal{T}_{\mu \cup \omega}^{\text{SStd}}(\mu \cup \alpha_j), \mathbf{v} \in \mathcal{T}^{\text{Std}}(\mu \cup \alpha_j), \text{ for } 1 \leq j \leq a \}.$$

*In particular,  $\text{Ind } S(\mu)$  has a filtration  $0 = I_0 \subset I_1 \subset \cdots \subset I_a = \text{Ind } S(\mu)$  such that  $I_j/I_{j-1} \cong S(\mu \cup \alpha_j)$ , for  $j = 1, \dots, a$ .*

*Proof.* That  $\text{Ind } S(\mu) \cong M(\mu \cup \omega)/N^\alpha$  is a special case of Theorem (3.6). The second claim follows from (2.3) by setting  $I_j = N^{\alpha_{j+1}}/N^\alpha$ , for  $0 \leq j < a$ . To prove that  $S(\mu \cup \alpha_j) \cong I_j/I_{j-1}$ , for  $1 \leq j \leq a$ , observe that the bijective map

$$S(\mu \cup \alpha_j) \longrightarrow I_j/I_{j-1}; m_{\mathbf{s}} \mapsto m_{(\mathbf{T}^\mu \cup \alpha_j)_{\mathbf{s}}} + I_{j-1}, \quad \text{for } \mathbf{s} \in \mathcal{T}^{\text{Std}}(\mu \cup \alpha_j),$$

commutes with the action of  $\mathcal{H}_{n+1}$ . (Here,  $\mathbf{T}^\mu = \mu(\mathbf{t}^\mu)$  is the unique semistandard  $\mu$ -tableau of type  $\mu$ .)  $\square$

**(3.8). Remark.** Maintain the notation of Theorem (3.6) and define integers  $a_i$  and multipartitions  $\lambda_{i,j}$  by writing  $\{\lambda_{i,1} \triangleright \cdots \triangleright \lambda_{i,a_i}\} = \{\text{Shape}(\mathbf{U}) \mid \mathbf{U} \in \mathcal{T}_{\mu \cup \omega}^{\text{SStd}}(\mathbf{S}_i)\}$ , for  $i = 1, \dots, m$ . Theorem (3.6) then implies, just as in the proof of Corollary (3.7), that  $M(\mu \cup \omega)$  has a Specht filtration

$$0 \subset I_{1,1} \subset \cdots \subset I_{1,a_1} \subset I_{2,1} \subset \cdots \subset I_{m,a_m} = M(\mu \cup \omega),$$

with  $I_{i,a}/I_{i,a}^< \cong S(\lambda_{i,a})$ , where  $I_{i,a}$  is the submodule of  $M(\mu \cup \omega)$  with basis

$$\{ m_{\mathbf{U}\mathbf{v}} \mid \mathbf{U} \in \mathcal{T}_{\mu \cup \omega}^{\text{SStd}}(\lambda_{j,b}), \mathbf{v} \in \mathcal{T}^{\text{Std}}(\lambda_{j,b}) \text{ where } j < i, \text{ or } j = i \text{ and } b \leq a \}$$

and where  $I_{i,a}^< = I_{i,a-1}$  if  $a > 1$ ,  $I_{i,1}^< = I_{i-1,a_{i-1}}$  if  $i > 1$  and  $I_{1,1}^< = 0$ .

Fred Goodman has pointed out that this filtration of  $M(\mu \cup \omega)$  is, in general, different to that given by (2.3) because the order in which the Specht modules appear does not have to be compatible with the dominance ordering—note, however, that the Specht modules in each ‘layer’  $N_i/N_{i-1}$  are totally ordered by dominance. For example, suppose that  $\ell = 1$  and let  $\mu = (3^2, 1)$  so that  $\mu \cup \alpha = (4, 3, 1)$  and  $\mu \cup \omega = (3^2, 1^2)$ . Then

$$\mathbf{U} = \begin{array}{|c|c|c|c|} \hline 1 & 1 & 1 & 2 \\ \hline 2 & 2 & & \\ \hline 3 & 4 & & \\ \hline \end{array}$$

is a semistandard  $\nu$ -tableau of type  $\mu \cup \omega$ , where  $\nu = (4, 2^2)$ . (As  $\ell = 1$  we can label semistandard tableaux with the integers  $1, \dots, n$ .) However,  $\mu \cup \alpha \triangleright \nu$  even though  $\nu \neq \mu \cup \beta$  for any addable node  $\beta$  of  $\mu$ .

As induction and restriction are both exact functors the main result of this note, together with [2, Prop. 1.9] (and the corresponding argument for the degenerate case), shows that the full subcategory of  $\mathcal{H}_n$ -**mod** which consists of modules which have a Specht filtration is closed under induction and restriction.

**(3.9). Corollary.** *Suppose that  $M$  has a Specht filtration. Then the modules  $\text{Res } M$  and  $\text{Ind } M$  both have Specht filtrations.*

In [17, Theorem 3.6] and [6, Theorem 4.6] it is shown that for each multipartition  $\mu \in \Lambda_{\ell,n}^+$  there exists an indecomposable  $\mathcal{H}_n$ -module  $Y(\mu)$ , a **Young module**, such that

$$M(\mu) \cong Y(\mu) \oplus \bigoplus_{\lambda \triangleright \mu} Y(\lambda)^{\oplus c_{\lambda\mu}}$$

for some non-negative integers  $c_{\lambda\mu}$ . Each Young module  $Y(\mu)$  has a Specht filtration. Therefore, by Corollary (3.9),  $\text{Res } Y(\mu)$  and  $\text{Ind } Y(\mu)$  both have Specht filtrations.



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SCHOOL OF MATHEMATICS AND STATISTICS F07, UNIVERSITY OF SYDNEY, NSW 2006, AUSTRALIA.  
*E-mail address*: a.mathas@usyd.edu.au